Zusammenfassung DSP

Introduction

Classification of signals:
- real / complex
- multichannel / -dimensional
- Continious-time / -valued
- Discrete-time / -valued

Digital processing of analog signals:

$$x_a(t) \rightarrow \underbrace{\boxed{\text{Sampling}} \xrightarrow{x(n)} \boxed{\text{Quantization}}}_{\boxed{\text{ADC}}} \xrightarrow{x_q(n)} \boxed{\text{DSP}} \xrightarrow{y_q(n)} \boxed{\text{DAC}} \xrightarrow{y_a(t)}$$

Sampling: $x(n) = x_a(T_n) \quad n \in \mathbb{Z}$

uniform: $T_n = n \cdot T + t_0$ $\quad$ T ... sampling period ($F_s = \frac{1}{T}$ ... sampling frequency)

Quantization: $x_q(n) = Q[x(n)]$ $\quad$ $Q[\sim]$ ... quantization operator (using b bits per value) ↳ 2^b possible values

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Discrete-time signals and systems

Discrete-time signals: Manipulations:
- (i) delay: $x(n) \rightarrow x(n-k)$
- (ii) advance: $x(n) \rightarrow x(n+k)$
- (iii) time reversal: $x(n) \rightarrow x(-n)$
- (iv) decimation: $x(n) \rightarrow x(m \cdot n)$
- (v) scaling: $x(n) \rightarrow a \cdot x(n)$
- (vi) addition: $x_1(n) + x_2(n)$
- (vii) multiplication: $x_1(n) \cdot x_2(n)$

$k, m \in \mathbb{N}$

Discrete-time systems: ~~Classification~~ $y(n) = T(x(n))$

Classifications:
- Memoryless: $y(n) = f(x(n))$
- Causal: $y(n) = f(x(n), x(n-1), \dots)$ $\quad$ [Anticausal: $y(n) = f(x(n), x(n+1), \dots)$]
- Linear: $y_i(n) = T(x_i(n)) \rightarrow T\left(\sum_i c_i x_i(n)\right) = \sum_i c_i y_i(n)$

  Properties: $\cdot\ T(c \cdot x(n)) = c \cdot T(x(n))$ $\quad$ $\cdot\ T(0) = 0$

  $\cdot\ T\left(\sum_i x_i(n)\right) = \sum T(x_i(n))$
- Time-Invariant: $T(x(n-k)) = y(n-k)$
- BIBO-stable: $|x(n)| \leq B_x < \infty \rightarrow |y(n)| \leq B_y < \infty \quad \forall n, x(n)$

Analysis of linear system in time domain

Linear time-variant systems:
- impulse response: $h(n,k) = T(\delta(n-k))$ (mathematical description of a system)
- Decompose $x(n)$: $x(n) = \sum_k x(k)\, \delta(n-k) \rightarrow y(n) = \sum_k x(k)\, h(n,k)$

LTI systems: Convolution sum: $(x * h)(n) := \sum_k x(k)\, h(n-k) \rightarrow y(n) = (x * h)(n)$

Properties:
- $(x * h)(n) = (h * x)(n)$ $\quad$ $N_1 \cdot N_2$ multipl.
- $(x * h_1) * h_2 = x * (h_1 * h_2)$ $\quad$ $(N_1 - 1) \cdot N_2$ addition
- $x * h_1 + x * h_2 = x * (h_1 + h_2)$

Stability:
- LTI system BIBO-stable $\leftrightarrow \sum_n |h(n)| < \infty$
- $\sum_n |h(n)| < \infty \rightarrow h(n) \xrightarrow{n \to \infty} 0$

[FIR always BIBO-stable
IIR can be stable or not

Linear const.-coeff. difference equation

Difference equations: General: $\underbrace{a_0 y(n) + a_1 y(n-1) + \dots + a_N y(n-N)}_{\alpha(D) \cdot y(n)} = \underbrace{b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M)}_{\beta(D) \cdot x(n)}$

- linear diff. equ. → linear system
- a_i, b_i constant → time-invariant
- causal system → $a_0 \neq 0$ ($=1$ after normalization)
- degree: N

Nonrecursive & recursive system:

Nonrecursive: $a_1 = \dots = a_N = 0$, $\alpha(D) = a_0 = 1 \Rightarrow y(n) = b_0 x(n) + \dots + b_M x(n-M)$

Purely recursive: $b_1 = \dots = b_M = 0$, $\beta(D) = b_0 \Rightarrow y(n) + a_1 y(n-1) + \dots + a_N y(n-N) = b_0 x(n)$

General recursive: $y(n) + \underbrace{\sum_{i=1}^{N} a_i y(n-i)}_{\text{recursive part}} = \underbrace{\sum_{i=0}^{M} b_i x(n-i)}_{\text{nonrecursive part}}$

Solution of diff equ.: (i) recursive computation
(ii) cooking receipe

Impulse response: $h(n) = T(\delta(n))$, $\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$

Computing: 1) $h_{hom}(n) = \dots$

2) $h_{part}(n) = 0 \quad \forall\, n \geq N$ if $N > M$

3) $h(n) = h_h(n) + h_p(n) = \begin{cases} c_1 z_1^n + \dots + c_N z_N^n \\ (c_1 + c_2 n + \dots + c_m n^{m-1}) z_1^n + \dots \end{cases}$

4) zero initial conditions $\to c_1, \dots$

Stability: • BIBO-stable $\Leftrightarrow \sum_n |h(n)| < \infty$

• Causal system stable $\Leftrightarrow |z_i| \underset{(>)}{<} 1 \quad \forall i$ (anticausal)

Z-transform and it's applications

z-transform & it's ROC

• Def: $X(z) = \sum_{\mathbb{Z}} x(n) z^{-n}$, $z \in \mathbb{C}$ $\quad ROC(X(z)) = \{z \,;\, |X(z)| < \infty\}$

• $X(z)$ is not unique, but $X(z)$ + ROC is unique for $x(n)$

inverse z-transform: $x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$

Properties $x(n) \xrightarrow{z} X(z)$, ROC: $r_1 < |z| < r_2$:

• time reversal: $x(-n) \xrightarrow{z} X(\frac{1}{z})$; ROC: $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

• linear: $x_i(n) \xrightarrow{z} X_i(z) \Rightarrow \sum c_i x_i(n) \xrightarrow{z} \sum c_i X_i(z)$ ROC: at least $\cap ROC_i$

• shift: $x(n-k) \xrightarrow{z} z^{-k} X(z)$

• stretch: $a^n x(n) \xrightarrow{z} X(\frac{z}{a})$; ROC: $r_1 \cdot |a| < |z| < r_2 \cdot |a|$

• differentiate: $n \cdot x(n) \xrightarrow{z} -z \frac{dX(z)}{dz}$

• moment: $\sum_{n \in \mathbb{Z}} n \cdot x(n) \xrightarrow{z} -z \cdot \frac{dX(z)}{dz}\Big|_{z=1}$

• convolution: $x_1(n) * x_2(n) \xrightarrow{z} X_1(z) \cdot X_2(z)$; ROC: at least $ROC_1 \cap ROC_2$

• initial value: $x(n)$ causal: $x(0) = \lim_{z \to \infty} X(z)$ $\quad x(n)$ anticausal: $x(0) = X(0)$

• $x^*(n) \xrightarrow{z} X^*(z^*)$, no change for ROC

Rational z-trafo: • Def: $X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$ with zero's z_i ($B(z_i)=0$) and poles p_i ($A(p_i)=0$)
(Always max(M,N) poles and zeros)

• no poles in ROC

• pole-zero-plot determines uniquely $X(z)$ (except for a scaling: $X(z) = \frac{b_0}{a_0} \frac{(z-z_1)\cdots}{(z-p_1)\cdots}$

Transfer function of LTI systems

• LTI: $y(n) = (h * x)(n)$

• transfer/system function: $H(z) = \mathcal{Z}(h(n)) \Leftrightarrow H(z) = \frac{Y(z)}{X(z)}$

• $H(z) = \frac{\beta(z^{-1})}{\alpha(z^{-1})} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$ $\Rightarrow$ diff. equation $\Leftrightarrow$ rational transfer function

• Classifications: • Nonrecursive (FIR): $N=0$, $\alpha(z^{-1}) = a_0 = 1 \Rightarrow H(z) = \beta(z^{-1}) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M} = \frac{b_0}{z^M}(z^M + \dots + \frac{b_M}{b_0})$

*) M zeros of $H(z) \hat{=}$ roots of $\beta(z^{-1})$
*) M trivial poles at $z=0$
} all-zero-system

• purely recursive system (IIR): $M=0$, $\beta(z^{-1}) = b_0 \Rightarrow H(z) = \frac{b_0}{\alpha(z^{-1})} = \frac{b_0 z^N}{z^N + a_1 z^{N-1} + \dots + a_N}$

*) N poles of $H(z) \hat{=}$ roots of $\alpha(z^{-1})$
*) N trivial poles at $z=0$
} pole-zero-system

• general recursive: $N>0$, $M>0 \Rightarrow$ pole-zero-system

• Stability: System BIBO stable $\Leftrightarrow$ ROC of $H(z)$ containing the unit circle $|z|=1$

• Pole-zero cancellation: $H(z)$ rational, poles p_i, zeros z_i; if $p_i = z_j \Rightarrow$ pole-zero-cancellation
*) reduced polynom degree in $\alpha(z^{-1})$, $\beta(z^{-1})$
*) changed ROC & stability

Transfer function of LTI-system

• Inverse z-transform: (i) Residue theorem: $x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$, $C \in ROC$ (not practical)

(ii) Compare $X(z)$ with $\sum_n x(n) z^{-n}$

(iii) Partial fraction expansion

• Properties : Same as Z{} except for : * delay property: $x(n-k) \xrightarrow{Z^+} z^{-k}\left\{X^+(z) + \sum_{n=-k}^{-1} x(n) z^{-n}\right\}$

* advance property: $x(n+k) \xrightarrow{Z^+} z^{k}\left[X^+(z) - \sum_{n=0}^{k-1} x(n) z^{-n}\right]$

• Solution of a diff. equ. with initial values:

given: $\alpha(D) y(n) = \beta(D) x(n)$, $x(n)$ known for $n \geq 0$, initial values $x(-1), \dots, x(-M), y(-1), \dots, y(-N)$

↳ $Z^+(\alpha(D) y(n)) = a_0 \cdot Y^+(z)$
$+ a_1 z^{-1}\left[Y^+(z) + \sum_{n=-1}^{-1} y(n) z^{-n}\right]$
$\vdots$
$+ a_N z^{-N}\left(Y^+(z) + \sum_{n=-N}^{-1} y(n) z^{-n}\right)$

$= \alpha(z^{-1}) Y^+(z) + \gamma(z^{-1}, y(-1), \dots, y(-N))$
$\stackrel{!}{=} \beta(z^{-1}) X^+(z) + \sigma(z^{-1}, x(-1), \dots, x(-M))$

$\Rightarrow Y^+(z) = \frac{\beta(z^{-1})}{\alpha(z^{-1})} X^+(z) + \frac{\sigma(z^{-1}) - \gamma(z^{-1})}{\alpha(z^{-1})}$

Frequency analysis of signals & systems

Period. signals & frequency: $x_a(t) = A \cdot \sin(\Omega t + \varphi) \xrightarrow[t = n \cdot T]{\text{uniform sampling}} x(n)$

$T > 0$: sampling period [s]
$F_s = \frac{1}{T}$: sampling frequ. [Hz]
$f = \frac{F}{F_s} = F \cdot T$: normalized frequ.
$\omega = 2\pi f$: normalized angular frequ.

↳ $x(n) = A \cdot \sin(2\pi F n T) = A \sin(2\pi f n + \varphi) = A \sin(\omega n + \varphi)$

Nyquist condition: $F_s > 2 F_{max}$

Frequ. analysis of cont.-time signals: Fourier series: $x(t) = \sum_k c_k e^{jk\Omega_0 t}$ with $\Omega_0 = \frac{2\pi}{T}$ and $c_k = \frac{1}{T} \int_{(T)} x(t) e^{-jk\Omega_0 t} dt$

Fourier transform: $X(j\Omega) = \int_{\mathbb{R}} x(t) e^{-j\Omega t} dt$

Frequ. analysis of discrete-time signals: ~~[illegible]~~

Fourier series: $x(n+N) = x(n) \; \forall n$ ($N > 0$: period)

Fourier series: $x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi \frac{k}{N} n}$ with $\frac{k}{N} = f_k$

Fourier coeff.: $c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j 2\pi \frac{k}{N} n}$ $(0 \leq k \leq N-1)$

Properties: *) $c_k = c_{k+N}$ $\forall k$ → $0 \leq k \leq N-1$
↳ $0 \leq f < 1$ fundamental frequ. range

build symmetric Σ !

*) Parseval's relationship:

average power of $x(n)$:
$\frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |c_k|^2$ power density spectrum

Fourier transform: $x(n)$ not periodic $\boxed{X(e^{j\omega}) = \sum x(n) e^{-j\omega n}}$

$\Leftrightarrow x(n) \xrightarrow{F} X(e^{j\omega})$ ⟷ $x(n) = \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \frac{d\omega}{2\pi}$

Existence: $X(e^{j\omega})$ exists if $\sum |x(n)| < \infty$

$F \leftrightarrow Z$: • $X(z) = \sum_n x(n) z^{-n}$, ROC

↳ ROC contains $|z| = 1$: $X(e^{j\omega}) = X(z)|_{z = e^{j\omega}}$

• $X(e^{j\omega}) = X(e^{j(\omega + 2\pi)})$

↳ $0 \leq \omega < 2\pi$
$0 \leq f < 1$
$0 \leq F = f \cdot F_s < F_s$

Sampling theorem: $x_a(t) \xrightarrow{F} X_a(j\Omega)$
$\downarrow t = nT$
$x(n) \xrightarrow{F} X(e^{j\omega})$

$\rightarrow X(e^{j\Omega T}) = \frac{1}{T} \sum_k (j\Omega + jk\Omega_s)$ with $\Omega_s = \frac{2\pi}{T}$

bitrate: $F_s \cdot$ resolution

resolution : $\frac{\Delta x}{2^n}$ at n bit-A/D

Reconstruction of $x_a(t)$: $X_a(j\Omega) = \begin{cases} T \cdot X(e^{j\Omega T}) & |\Omega| < \Omega_{s/2} \\ 0 & \text{otherwise} \end{cases}$ $\Rightarrow x_a(t) = \int_{\mathbb{R}} X_a(j\Omega) e^{j\Omega t} \frac{d\Omega}{2\pi}$

F-Transform

Properties:

- time reversal: $x(-n) \xrightarrow{z} X(z^{-1})$, $\xrightarrow{F} X(e^{-j\omega})$
- conj. complex: $x^*(n) \xrightarrow{z} X^*(z^*)$, $\xrightarrow{F} X^*(e^{-j\omega})$
- periodic: $X(e^{j\omega}) = X(e^{j\omega 2\pi f})$
- linear: $\sum_i c_i x_i(n) \xrightarrow{z} \sum_i c_i X_i(z)$, $\xrightarrow{F} \sum_i c_i X_i(e^{j\omega})$
- differential: $(-jn)^m x(n) \xrightarrow{F} \frac{d^m}{d\omega^m} X(e^{j\omega})$
- convolution: $x_1(n) * x_2(n) \to X_1(z) \cdot X_2(z)$
- Parseval: $\sum x_1(n) x_2^*(n) = \int_{-\pi}^{\pi} X_1(e^{j\omega}) X_2^*(e^{j\omega}) \frac{d\omega}{2\pi}$ ($\to$ Energy: $E = \sum_n |x(n)|^2 = \int_{-\pi}^{\pi} \underbrace{|X(e^{j\omega})|^2}_{\text{spectrum}} \frac{d\omega}{2\pi}$)

	$x(n)$	$X(e^{j\omega})$
real + even	x	x
~~imag + even~~		
odd + real	x	x
odd + imag	x	x

- shift in frequ.: $a^n x(n) \xrightarrow{z} X(\frac{z}{a})$ $a \in \mathbb{C}$; $e^{j\omega_0 n} x(n) \xrightarrow{F} X(e^{j(\omega-\omega_0)})$
- shift in time: $x(n-n_0) \xrightarrow{z} z^{-n_0} X(z)$, $\xrightarrow{F} e^{-j\omega n_0} X(e^{j\omega})$
- moment: $\sum_{n=-\infty}^{\infty} n^m x(n) = j^m \frac{d^m}{d\omega^m} X(e^{j\omega})|_{\omega=0}$
- multiplication: $x_1(n) \cdot x_2(n) \xrightarrow{F} \int_{-\pi}^{\pi} X_1(e^{j\lambda}) X_2(e^{j(\omega-\lambda)}) \frac{d\lambda}{2\pi}$

Frequ. analysis of LTI system

Frequ. response: $H(j\omega) = F(h(n)) = A(\omega) \cdot e^{j\cdot\Theta(\omega)}$ with $A(\omega)$: Amplitude response, $\Theta(\omega)$: phase "

$\tau_g(\omega) = -\frac{d\Theta(\omega)}{d\omega}$ ... group delay

Physical interpretation: $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$

Relation to $H(z)$: *) $H(j\omega) = H(z)|_{z=e^{j\omega}}$ if $|z|=1 \in ROC$

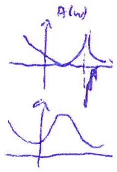
*) $A^2(\omega) = H(z) \cdot H^*(\frac{1}{z^*})|_{z=e^{j\omega}}$

Digital Filters:

Ideal filters: $Y(j\omega) = H(j\omega) \cdot X(j\omega) = A_0 \cdot e^{-j\omega n_0} \cdot X(j\omega) \Rightarrow y(n) = A_0\, x(n-n_0)$

Method of pole-zero-placement: $H(z) = C \cdot \frac{\sum b_i z^i}{\sum_i a_i z^i}$

Poles / Zeros on unit circle: " near " :



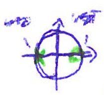
linear phase: FIR possible, IIR never

FIR: $y(n) = \sum b_i x(n-i)$ (always stable)

Rules:
- all poles inside $|z|=1$ → causal, stable
- poles in passband → large $A(\omega)$
- zeros in stopband → small/zero $A(\omega)$
- $h(n)$ real → complex-conjugate pairs

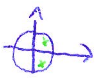
Low pass:
- poles around $\omega=0$
- zeros around $\omega=\pi$

High pass:
- poles around $\omega=\pi$
- zeros around $\omega=0$

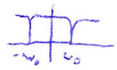


(HP from LP: $\omega = \omega + \pi$ bzw. $h_{HP}(n) = (-1)^n h_{LP}$

Band pass: compl. conjugate poles



Notch filter:



$\omega_0 = 0 \to$ DC-notch

$H(z) = b_0 (1-e^{j\omega_0} z^{-1}) \ldots (1-e^{-j\omega_0} \ldots$

Comb filter: DC-notch $h_n \xrightarrow[\text{by factor } N]{\text{up sampling}}$ comb filter $h_c(n) = \begin{cases} h_n(\frac{n}{N}) & \text{if } n=k\cdot N \\ 0 \end{cases}$ i.e.

Digital Oscillators: instable digital filter

Linear phase FIR filter: $\Theta(\omega) = -n_0 \omega + \Theta_0$ ... linear phase

$\tau_g = -\frac{d\Theta}{d\omega} = n_0$ ... group delay

(always possible for FIR filters, never for IIR)

*) symmetric impulse response: $h(n) = h^*(N-n)$ ↳ $\Theta(\omega) = -\frac{N}{2}\omega$

*) antisymmetr. " : $h(n) = -h^*(N-n) \Rightarrow \Theta(\omega) = -\frac{N}{2}\omega + \pi$

Allpass filters: $A(\omega) = A_0 \neq 0\ \forall \omega$

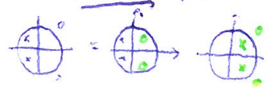
poles at $p = r e^{j\varphi}$, zeros $z = \frac{1}{r} e^{j\varphi}$

$H(z) = \frac{z^{-1} - p^*}{1 - p z^{-1}}$

Minimum-phase filters $|n_i| < 1\ \forall i$

↳ always a causal stable inverse filter

$H(z) = H_{min}(z) \cdot H_{all}(z)$:



Discrete Fourier transform

Frequency domain sampling: $X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_n x(n) e^{-j\omega n} \quad (0 \le \omega \le 2\pi)$

$X(k) := X(e^{j\omega_k}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi}{N}kn}$ ... uniform frequ. sampling

$\Delta\omega = \omega_k - \omega_{k-1} = \frac{2\pi}{N}$ ... sampling intervall

$X(k) = \ldots = \sum_{n=0}^{N-1} \Big(\underbrace{\sum_l x(n+lN)}_{x_p(n) = x_p(n+N)} \Big) e^{-j\frac{2\pi}{N}kn}$

$x_p(n) = x_p(n+N) \rightarrow$ Fourier series: $x_p(n) = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$

$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} X(k)$

Nyquist: $N \ge$ length of $x(n)$

Discrete Fourier transform:

$N = L$ (length of $x(n)$):

$DFT_N: X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x(n) W_N^{kn}$ with $W_N = e^{-j\frac{2\pi}{N}}$ ... DFT kernel

inverse DFT: $x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) (W^*)^{kn}$

(if $N > L$: $x(0), \ldots, x(L-1), 0, \ldots, 0$ $\hat{=}$ zero padding)

DFT as linear transform:

$\underline{X} = \underline{\underline{W}} \cdot \underline{x} \iff \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ 1 & W_N & \cdots & W_N^{N} \\ \vdots & & & \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)N} \end{bmatrix} \begin{bmatrix} x(0) \\ \vdots \\ x(N-1) \end{bmatrix}$

Properties of W_N: $|W_N| = 1 \qquad W_N^{N/2} = -1$

$W_N^N = 1 \qquad W_N^{-1} = W_N^*$

Properties of $\underline{\underline{W}}$: $\underline{\underline{W}}^T = \underline{\underline{W}}$ ... symmetric

$\underline{\underline{W}}^{-1} = \frac{1}{N}\underline{\underline{W}}^*$

$\underline{\underline{W}}^H = (\underline{\underline{W}}^T)^* = (\underline{\underline{W}}^*)^T = \underline{\underline{W}}^*$ ... hermitian

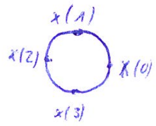
$\frac{1}{\sqrt{N}} \cdot \underline{\underline{W}}$ ... unitary matrix with orthog. row/column vectors

Properties of DFT:

Circular convolution: $x_1 \circledast x_2 = \sum_{m=0}^{N-1} x_{p_1}(m)\, x_{p_2}(n-m) \xrightarrow{DFT} X_1(k) X_2(k)$

length: $N_1 + N_2 - 1$

↳ Circular rotation:



$n \rightarrow -n$ ... reversal

$\rightarrow x_p(n-1)$ ... shift

Circular vs. linear: $x_1 \circledast x_2 \ne x_1 * x_2$

↳ Nyquist condition for output signal → zero padding

Fast Fourier transform:

Basic idea: · exploit properties of DFT kernel ($W_N = e^{-j\frac{2\pi}{N}}$): $*)\ W_N^{k+N} = W_N^k$

$*)\ W_N^{k+\frac{N}{2}} = -W_N^k$

$*)\ W_N^{2k} = W_{N/2}^k$

N^2 multipl.
$N(N-1)$ addition
(FFT: $N \log_2 N$)

· decompose DFT_N into smaller size DFT's: $N = \begin{cases} r_1 \cdot r_2 \cdots r_d & \rightarrow \text{mixed-radix FFT} \\ r^d & \rightarrow \text{radix-r FFT} \\ 2^u & \rightarrow \text{radix-2 FFT} \end{cases}$

Radix-2-decimation-in-time FFT:

decimation in time: decompose $x(0), \ldots, x(N-1)$ into: $\begin{cases} x(0), x(2), \ldots, x(N-2) \rightarrow DFT_{N/2} \\ x(1), x(3), \ldots, x(N-1) \rightarrow DFT_{N/2} \end{cases}$

$X(k) = \ldots = \underbrace{\sum_{n=0}^{N/2-1} x(2n) W_{N/2}^{kn}}_{F_1} + \underbrace{\sum_{n=0}^{N/2-1} x(2n+1) W_{N/2}^{kn}}_{F_2} \cdot W_N^k$ ... periodic with periode $\frac{N}{2}$

(i) compute $2 \times DFT_{N/2} \rightarrow F_1, F_2$

$\swarrow$ $X(k) = F_1(k) + F_2(k) W_N^k$

(ii) compute F_1, F_2 by $2\ DFT_{N/4} \rightarrow 4 \times DFT_{N/4}$ etc.

Complexity: for $N = 2^d$: *) d decomposition steps $\Rightarrow$ d stages

*) $\frac{N}{2}$ "·", N "±" $\Rightarrow$ $\frac{N}{2}\log_2 N$ "·", $N\log_2 N$ "±"

*) $\frac{N}{2}$ butterflies per stage

but: reversal ordering

Radix-2 decimation-in-frequency FFT:

(i) $f_1(n) = x(n) + x(n+\frac{N}{2})$

$f_2(n) = (x(n) - x(n+\frac{N}{2}))W_N^n$

(ii) $X(2k) = DFT_{N/2}(f_1(n))$

$X(2k+1) = DFT_{N/2}(f_2(n))$

· complexity & like dec.-in-time

· also reversal ordering

Relationship between both FFT's: change flow direction of signals (input ⇔ output, · ⇔ +)

Radix-r decimation-in-time FFT:

decompose $x(n)$ into

$x(0)$ $x(r)$ ... $x(N-r)$ $\Rightarrow DFT_{N/r}$

$x(1)$ $x(r+1)$ ... $x(N-r+1)$

$\vdots$

$x(r-1)$ $x(2r-1)$ ... $x(N-1)$

$\Rightarrow$ *) r $DFT_{N/r}$

*) $d = \log_r N$ stages

*) radix-r butterflies ($\frac{N}{r}$ times)

DFT of 2 real signals: idea: $x(n) = x_1(n) + jx_2(n)$ $\Rightarrow$ $X_1(k) = \frac{1}{2}(X(k) + X^*(N-k))$

$X_2(k) = \frac{1}{2j}(X(k) - X^*(N-k))$

DFT of 1 real signal: given: $y(n) \in \mathbb{R}$ length $2N$

$x_1(n) = y(2n)$, $x_2(n) = y(2n+1)$ $\Rightarrow x = x_1 + jx_2$

↳ $X_1(k) = \frac{1}{2}(X(k) + X^*(N-k))$

$X_2(k) = \frac{1}{2j}(\text{''} - \text{''})$

$\Rightarrow$ $Y(k) = X_1(k) + W_{2N}^k X_2(k)$

$Y(k+N) = \text{''} - \text{''}$

Overlap-Add ($N < \underbrace{N_1}_{>N} + N_2 - 1$)

↳ partition x_1 in blocks of length M

↳ $N = M + N_2 - 1$

Lösen d. Zustandsgleichung

$\circ \nabla x(t),\ -\infty < t < \infty \nabla$

*) Komplexe Ebene: $\nabla \underline{\underline{H}}(p) = \underline{\underline{C}}(p\underline{\underline{I}}-\underline{\underline{A}})^{-1}\underline{\underline{B}} + \underline{\underline{D}}$

$\hookrightarrow = \dfrac{[adj_{ij}]}{\det(p\underline{\underline{I}}-\underline{\underline{A}})}$ mit $adj_{ij} = (-1)^{i+j}\det[\ ]$

$x(t),\ t>0$:

$$\underline{Y}^+(p) = \underline{\underline{H}}(p)\,\underline{X}^+(p) + \underline{\underline{C}}(p\underline{\underline{I}}-A)^{-1}\cdot\underline{z}_0$$

*) Zeitbereich: $(t>0)$

$$\underline{z}(t) = \underline{\underline{\phi}}(t^{t-t_0})\cdot\underline{z}_0 + \int_{0/t_0}^{t}\underline{\underline{\phi}}(t-\tau)\,\underline{\underline{B}}\,\underline{x}(\tau)\,d\tau$$

$$y(t) = \underline{\underline{C}}\,\underline{z}(t) + \underline{\underline{D}}\,\underline{x}(t) = \underline{\underline{C}}\,\underline{\underline{\phi}}^{t-t_0}\underline{z}_0 + \int_{0/t_0}^{\infty} h(t-\tau)\,x(\tau)\,d\tau$$

$$\underline{\underline{h}}(t) = \underline{\underline{C}}\cdot\underline{\underline{\phi}}(t)\,\underline{u}(t)\,\underline{\underline{B}} + \underline{\underline{D}}\,\delta(t)$$

*) $\underline{\underline{\phi}}(t) \hat{=}$ Zustandsübergangs-~~matrix~~ $= e^{\underline{\underline{A}}t}$

*) Falls $\underline{\underline{A}}$ invertierbar:

$$\underline{\underline{\phi}}(t) = \underline{\underline{V}}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}\underline{\underline{V}}^{-1}$$

*) sonst: $\mathcal{L}^{-1}(p\cdot\underline{\underline{I}}-\underline{\underline{A}})^{-1}$

Eigenschaften: $\underline{\underline{\phi}}(0) = \underline{\underline{I}}$

$\underline{\underline{\phi}}^{-1}(t) = \underline{\underline{\phi}}(-t)$

$\underline{\underline{\phi}}^{k}(t) = \underline{\underline{\phi}}(kt)$

$\underline{\underline{\phi}}(t_1)\cdot\underline{\underline{\phi}}(t_2) = \underline{\underline{\phi}}(t_1+t_2)$

$$\arctan\frac{y}{x} = \begin{cases} \arctan\frac{y}{x} & x>0 \\ \text{"} + \pi & x<0,\ y\geq 0 \\ \text{"} - \pi & x<0,\ y<0 \\ \frac{\pi}{2} & x=0,\ y>0 \\ -\frac{\pi}{2} & x=0,\ y<0 \\ 0 & x=0,\ y=0 \end{cases}$$